Uniqueness of the Fock quantization of Dirac fields with unitary dynamics

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Motivation

- Dirac (fermion) fields describe realistic matter contents in Physics: cosmology, condensed matter (e.g. graphene)...

- Infinite ambiguity in their quantum description:
    - Choice of (Fock) representation of the CAR's.
- In non-stationary geometries, it is reasonable to require the dynamics to be unitary (quantum coherence).

- Additional criterion: invariant vacuum under the spatial isometries.

- **Result:** <u>unique</u> Fock representation of the CAR's.

Closed FLRW cosmology: Invariant vacua

Cosmological model

- FLRW cosmology, scale factor $\exp[\alpha(\eta)]$, spatial sections $\approx S^3$.

- Minimally coupled massive Dirac field, described by

  $\phi_A,\ \bar{\chi}_{A'} \qquad A=1,2;\ A'=1',2', \qquad$ Grassmann variables.

- Mode decomposition in terms of Dirac operator eigenspinors on S^3:

$$\left.\begin{array}{l} \phi_A \longrightarrow m_{np}(\eta), \bar{r}_{np}(\eta), \text{ eigenvalues } \pm\omega_n \\ \bar{\chi}_{A'} \longrightarrow t_{np}(\eta), \bar{s}_{np}(\eta), \text{ eigenvalues } \pm\omega_n \end{array}\right\} \begin{array}{l} \omega_n \sim n \in \mathbb{N} \\ p=1,\ldots,g_n \sim n^2 \end{array}$$

- Fock representation $\longrightarrow$ complex structure on space of initial data.

Isometry invariant vacua

- Isometry group $SO(4) \longrightarrow \mathrm{Spin}(4)$ on spinors of a chirality.

- Known relation between the Dirac operator eigenspaces on S^3 and the (unitary) irreducible representations of $\mathrm{Spin}(4)$.

- **Result:** any $\mathrm{Spin}(4)$-invariant Fock vacuum is characterized by particle annihilation and antiparticle creation variables:

$$a_{np}(\eta) = f_1^n(\eta)\, x_{np}(\eta) + f_2^n(\eta)\, \bar{y}_{np}(\eta),$$

$$b_{np}^{\dagger}(\eta) = g_1^n(\eta)\, x_{np}(\eta) + g_2^n(\eta)\, \bar{y}_{np}(\eta),$$

$$(x_{np}, y_{np}) = \begin{cases} (m_{np}, s_{np}) \\ (t_{np}, r_{np}) \end{cases}$$

$$|f_1^n|^2 + |f_2^n|^2 = e^{3\alpha} = |g_1^n|^2 + |g_2^n|^2, \qquad f_1^n \bar{g}_1^n + f_2^n \bar{g}_2^n = 0.$$

Unitary dynamics

Fermion dynamics

- First order Dirac equations, with $' := d/d\eta$

$$x_{np}' = \left(i\omega_n - \frac{3\alpha'}{2} \right) x_{np} - ime^{\alpha}\bar{y}_{np}, \quad \bar{y}_{np}' = -\left(i\omega_n + \frac{3\alpha'}{2} \right) \bar{y}_{np} - ime^{\alpha} x_{np}.$$

- Same second order equation for all the modes.
  - Known asymptotic behavior of its two independent solutions.

- The asymptotics of the evolution is known

$$x_{np}(\eta) = A_n(\eta, \eta_0)\, x_{np}(\eta_0) + B_n(\eta, \eta_0)\, \bar{y}_{np}(\eta_0),$$

$$\bar{y}_{np}(\eta) = \bar{A}_n(\eta, \eta_0)\, \bar{y}_{np}(\eta_0) - \bar{B}_n(\eta, \eta_0)\, x_{np}(\eta_0).$$

Unitary dynamics

- Fermion dynamics $\longrightarrow$ time-dependent Bogoliubov transformation:

$$a_{np}(\eta)=\alpha_n^f(\eta,\eta_0)a_{np}(\eta_0)+\beta_n^f(\eta,\eta_0)b_{np}^\dagger(\eta_0)$$

$$b_{np}^\dagger(\eta)=\alpha_n^g(\eta,\eta_0)b_{np}^\dagger(\eta_0)+\beta_n^g(\eta,\eta_0)a_{np}(\eta_0)$$

- The transformation is implementable as a unitary operator in the Fock space defined by $\{a_{np}(\eta_0), b_{np}^\dagger(\eta_0)\}$ if and only if

$$\sum_n g_n|\beta_n^f(\eta,\eta_0)|^2<\infty, \quad \sum_n g_n|\beta_n^g(\eta,\eta_0)|^2<\infty, \qquad \forall\, \eta.$$

- We know that $\omega_n \sim n \in \mathbb{N}$ and $g_n \sim n^2$.

Unitary dynamics: conditions

$$a_{np}(\eta) = f_1^n(\eta)\, x_{np}(\eta) + f_2^n(\eta)\, \bar{y}_{np}(\eta),$$

$$b_{np}^{\dagger}(\eta) = g_1^n(\eta)\, x_{np}(\eta) + g_2^n(\eta)\, \bar{y}_{np}(\eta).$$

- **Result:** Unitarily implementable dynamics if and only if:

$$\begin{pmatrix} f_1^n = \dfrac{m e^{5\alpha/2}}{2\omega_n} e^{iF^n} + \vartheta_{f,1}^n, & g_1^n = \bar{f}_2^n e^{iG^n}, \\ f_2^n = e^{iF^n} \sqrt{e^{3\alpha} - |f_1^n|^2}, & g_2^n = -\bar{f}_1^n e^{iG^n}, \end{pmatrix} \quad \forall \eta,$$

where $\vartheta_{f,1}^n$ is square summable, for an infinite subset of $\mathbb{N}$, whereas $f_1^n \leftrightarrow g_1^n$, $f_2^n \leftrightarrow g_2^n$, for the complementary subset.

Unitary dynamics: conditions

$$a_{np}(\eta)=f_1^n(\eta)\,x_{np}(\eta)+f_2^n(\eta)\,\bar{y}_{np}(\eta),$$

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where $\vartheta_{f,1}^n$ is square summable, for an infinite subset of $\mathbb{N}$, whereas $\underline{f_1^n \leftrightarrow g_1^n,\ f_2^n \leftrightarrow g_2^n,}$ for the complementary subset.

With respect to the first subset, particles $\leftrightarrow$ antiparticles.


Uniqueness

Uniqueness

- Reference representation that admits unitary dynamics:

$$f_1^n = \frac{m e^{5\alpha/2}}{2\omega_n}, \quad f_2^n = \sqrt{e^{3\alpha} - (f_1^n)^2}, \quad g_1^n = f_2^n, \quad g_2^n = -f_1^n, \qquad \forall\, n \in \mathbb{N}.$$

- Any other with the same convention of particles and antiparticles:

$$\tilde{f}_1^n = \frac{m e^{5\alpha/2}}{2\omega_n} e^{i \tilde{F}^n} + \vartheta^n_{\tilde{f},1}, \qquad \sum_{n \in \mathbb{N}} g_n \left|\vartheta^n_{\tilde{f},1}\right|^2 < \infty, \qquad \forall\, n \in \mathbb{N}, \qquad \text{etc.}$$

- Bogoliubov transformation between them:

$$\tilde{a}_{np} = \kappa_n^f a_{np} + \lambda_n^f b^\dagger_{np}, \qquad \lambda_n^f = \vartheta^n_{\tilde{f},1} + O(\omega_n^{-2}),$$

$$\tilde{b}^\dagger_{np} = \kappa_n^g b^\dagger_{np} + \lambda_n^g a_{np}, \qquad \lambda_n^g = -\bar{\vartheta}^n_{\tilde{f},1} e^{i\tilde{G}_n} + O(\omega_n^{-2})$$

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$$\left.\begin{aligned} \tilde{a}_{np} &= \kappa_n^f a_{np} + \lambda_n^f b^\dagger_{np}, & \lambda_n^f &= \vartheta^n_{\tilde{f},1} + O(\omega_n^{-2}), \\ \tilde{b}^\dagger_{np} &= \kappa_n^g b^\dagger_{np} + \lambda_n^g a_{np}, & \lambda_n^g &= -\bar{\vartheta}^n_{\tilde{f},1} e^{i\tilde{G}_n} + O(\omega_n^{-2}) \end{aligned}\right\} \quad \text{Unitary equivalence}$$

$$\omega_n \sim n \in \mathbb{N}, \quad g_n \sim n^2$$

Conclusions

- Combined criteria of invariance of the vacuum under the isometry group of the closed FLRW cosmology + unitary implementation of the dynamics $\longrightarrow$ unique Fock quantization of the Dirac field.

- Uniqueness attained given a convention of particles and antiparticles.

- The part of the dynamics that can be unitarily implementable is uniquely determined $\longrightarrow$ extraction of explicitly time-dependent functions from the dominant parts of the field.

- Generalization to flat spatial sections, and $2+1$ scenarios.